Periodic billiard orbits in right triangles II

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Abstract. Periodic billiard orbits are dense in the phase space of an irrational right triangle. A stronger pointwise density result is also proven.

1. Introduction

A billiard ball, i.e. a point mass, moves inside a polygon $P \subset \mathbb{R}^2$ with unit speed along a straight line until it reaches the boundary ∂P, then instantaneously changes direction according to the mirror law: "the angle of incidence is equal to the angle of reflection," and continues along the new line. If the trajectory hits a corner of the polygon, in general it does not have a unique continuation and thus by definition it stops there.

It is an open question if there exists a periodic billiard orbit in every polygon. None the less for certain classes of polygons one can exhibit the existence of many periodic orbits. In particular one can ask how dense the periodic orbits are. Most of the known results are about rational polygons, i.e. polygons for which the angles between the sides are rational multiples of π. The first result in this direction was that of H. Masur who showed that the directions of periodic orbits are dense in a rational polygon [Ma]. This was strengthened by Boshernitzan et al. who showed that for a rational polygon periodic orbits are dense in the phase space [BoGaKrTr]. In this article they also showed a pointwise density result in the configuration space: in a rational polygon P there exists a dense G_δ set $G \subset P$ such that for each point $p \in G$ the orbit of (p, θ) is periodic for a dense subset of directions $\theta \in \mathbb{S}^1$. Vorobets strengthened this result to show that the set G is also of full measure [Vo].

Recently I showed that there is an open set $\mathcal{O}$ of right triangles such that for each irrational $P \in \mathcal{O}$ the set of periodic billiard orbits is dense in the phase space [Tr]. The main result of this article is a twofold strengthening of this result. First of all I extend the result to all irrational right triangles.

Theorem 1. *Periodic orbits are dense in the phase space of any irrational right triangle.*

Remark that this density result also holds for rational right triangles [BoGaKrTr]. Next I strengthen this density to a pointwise density statement.

Theorem 2. *Suppose that P is any irrational right triangle. Then there exists an at most countable set $B \subset P$ such that for every $p \in P \backslash B$ the orbit of (p, θ) is periodic for a dense subset of directions $\theta \in \mathbb{S}^1$.*

Billiards in right triangles are well known to be equivalent to the motion of two elastic point masses on a segment (see for example [MaTa]). Theorem 2 tells us that except for an at most countable set B of initial positions $0 \le x_1 \le x_2 \le 1$ if $(x_1, x_2) \not\in B$ then the orbit of $(x_1, v_1), (x_2, v_2)$ is periodic for a dense set of velocities (v_1, v_2).

Somewhat surprisingly Theorem 2 is stronger than the result of Vorobets for rational polygons. There is a special class of rational polygons known as Veech polygons which are well studied, see for example [MaTa] for the definition. Combing known results on Veech polygons and the arguments of this article yields

Proposition 3. *In P is a Veech polygon then there exists an at most countable set $B \subset P$ such that for every $p \in P \backslash B$ the orbit of (p, θ) is periodic for a dense subset of directions $\theta \in \mathbb{S}^1$.*

Arithmetic or square tiled polygons form a subclass of Veech polygons [Zo]. Let $V(P)$ be the set of corners of P. By definition there are no periodic orbits passing through $p \in V(P)$. Thus Theorem 2 and Proposition 3 imply that B is not empty since $V(P) \subset B$. A simple geometric argument shows

Proposition 4. *If P is arithmetic then for every $p \in P \backslash V(P)$ the orbit of (p, θ) is periodic for a dense subset of directions $\theta \in \mathbb{S}^1$.*

Note that there is a unique continuation of the billiard orbit through vertices with angle π/n. Such a vertex is called regularisable. If we consider such orbits as defined by this continuation then Proposition 4 holds for $V(P)$ defined as the nonregularisable vertices.

2. Strategy

Instead of reflection a billiard trajectory in a side of P one can reflect P in this side and unfold the trajectory to a straight line. Using this build an invariant flat surface as follows. Unfold P to a rhombus R. Nest unfold R to a surface S consist of a countable union $\{R_n : n \in \mathbb{Z}\}$ of copies of R. The interiors of the R_n are disjoint, and parallel copies of the same side are identified by translation (see Figure 1 where parallel sides are labelled by an integer). Let α be the smaller angle of the triangle P. Consider orbits which start in a direction θ in R_0. Then the label $n \in \mathbb{Z}$ corresponds to the billiard orbits in R in the direction

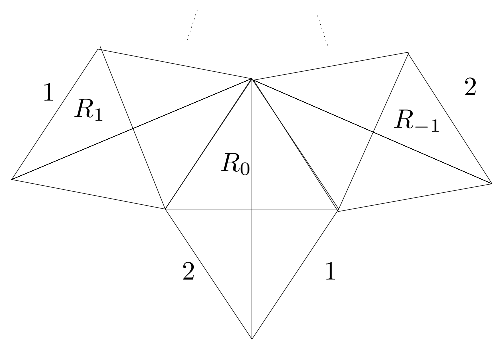

FIGURE 1. The invariant surface of an irrational triangle.

$\theta_n := \theta + 2n\alpha$. When we fix the initial direction θ we denote the surface by S_θ. Sometimes we will refer to the rhombus R_n simply as *level* n.

Consider the nonsingular orbit of a point $x := (s, \theta)$ and the sequence of labels $\{l(i) : i \in \mathbb{Z}\}$ corresponding to the sequence of labelled rhombi its orbit visits. The orbit is called *a forward escape orbit* if $\lim_{i\to\infty} l(i) = \infty$ and a *backward escape orbit* if $\lim_{i\to\infty} l(i) = -\infty$. Extend these notions to singular escape orbits, i.e. "orbits" which pass through one (or more) singularity which are defined by continuity from the left or right. The first step is to extend and simplify a result of [Tr].

Theorem 5. *In any irrational right triangle P, for each θ which is not parallel to the hypotenuse there is exactly one (possibly singular) forward escape orbit and exactly one (possibly singular) backward escape orbit on the surface S_θ.*

Call a direction *perpendicular* if it is perpendicular to one of the legs of the triangle or the hypotenuse. Let $L_\perp$ be the side of the triangle in question. An endpoint of $L_\perp$ is *good* if the perpendicular orbit of this point is twice perpendicular, and the second perpendicular hit is at an interior point of $L_\perp$. If $L_\perp$ is a leg is then call θ *end point good* if the endpoint which connects $L_\perp$ to the hypotenuse is good. If $L_\perp$ is the hypotenuse θ is *end point good* if both end points of $L_\perp$ are good.

Theorem 6. *Suppose P is an irrational right triangle. Fix an end point good perpendicular direction θ. Then all orbits on S_θ except for an at most countable collection of generalized diagonals and the unique forward and backward escape orbits are periodic.*

A special case of Theorem 6 was proven in [Tr], namely when $L_\perp$ is a leg of the triangle and the smaller irrational angle of P satisfies $\alpha \in (\pi/6, \pi/4)$.

Next verify that

Lemma 7. *Every irrational right triangle has an end point good perpendicular direction.*

The following Corollary follows immediately by combining Theorem 6 with Lemma 7.

Corollary 8. *Suppose that P is any irrational right triangle. Then, for one at least one of the three perpendicular directions θ, the invariant surface S_θ is foliated by periodic orbits except for an at most countable collection of orbits/generalized diagonals O_i.*

Proof of Theorem 1. The theorem follows from Corollary 8 and the fact that the surface S_θ is dense in the phase space of an irrational polygon. □

Turning to the pointwise result we need the following

Lemma 9. *Suppose P is an irrational polygon. Suppose that there exists a direction $\theta \in \mathbb{S}^1$ such that the invariant surface S_θ is foliated by periodic orbits except for an at most countable collection of orbits/generalized diagonals O_i. Then there exists an at most countable set $B \subset P$ such that for every $p \in P \backslash B$ the orbit of (p, θ) is periodic for a dense subset of directions $\theta \in \mathbb{S}^1$.*

Proof of Theorem 2. The theorem follows immediately by combining Corollary 8 and Lemma 9. □

3. Proofs

An orbit segment which begins and ends at a vertex of the polygon is called a generalized diagonal. A direction θ is called *simple* if there are no generalized diagonals on S_θ.

Proof of Theorem 5. First consider a simple direction θ. For simple directions the Theorem has already been proven in [Tr], however for completeness and to develop the proper notation I sketch the proof here. The billiard orbits in R_n enter either R_{n-1} or R_{n+1}. A single orbit separates these orbits into two sets, resp. R_n^- and R_n^+ which projectively can be thought of as intervals (see Figure 2). Call the orbit of such an interval a *strip*. All the results are obtained by considering compact regions. Let $K_N := R_0^+ \cup R_N^- \cup \cup_{1 \le n \le N-1} R_n$. Continue the backward orbit of the $N-1$ separatrices in K_N until the first time they hit the set $R_0^+ \cup R_N^-$. Projectively the set $R_0^+ \cup R_N^-$ consists of two interval. This pull back procedure splits it into $N+1$ intervals. Map each of these intervals until they return to the set $R_0 \cup R_N$. Note that this set is larger than $R_0^+ \cup R_N^-$. This covers K_N by a union of $N+1$ strips (with disjoint interiors).

Next consider the centers c_n of the rhombi R_n. The orbit c_n has a centrally symmetric code (labels of the rhombi it visits). In particular this implies that each of the $N-1$ centers in $K_N \backslash (R_0^+ \cup R_N^-)$ must be in different strips. By the central symmetry each of the strips which

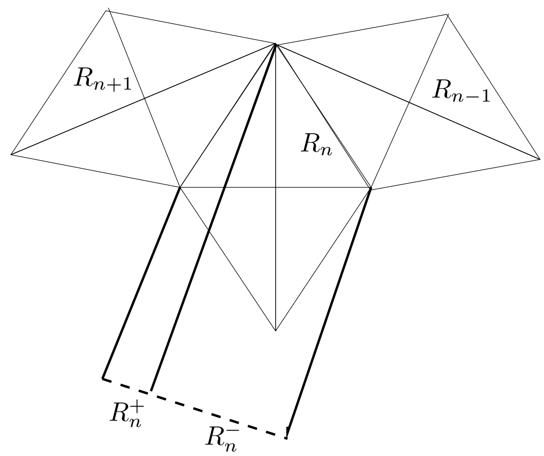

FIGURE 2. Projective view.

includes a center must start and stop on the same level (i.e. both on R_0 or both on R_N)

There are $2 = N+1-(N-1)$ exceptional strips. A simple argument (Lemma 10 of [Tr]) shows that for simple directions there must be an orbit from level 0 to level N and another from N to 0. Thus one of the exceptional strips, call it $E^+_{0,N}$, must start on level 0 and end on level N while the other, $E^-_{N,0}$, starts on level N and ends on level 0. Taking the intersection $\cap_{N\ge 0} E^+_{N,0}$ yields (the forward orbit of) a unique forward escape orbit. We can replace level 0 by an arbitrary negative integer to see that the forward escape orbit does not depend on level 0, i.e. for each $n \in \mathbb{Z}$ the image of the forward escape orbit on level n is the forward escape orbit on level $n+1$. Similarly $\cap_{N\ge 0} E^-_{0,N}$ yields a unique backward escape orbit.

In [Tr] the above argument was extended to directions perpendicular to one of the legs of the triangle formally under the condition that the angle α of the triangle satisfies $\alpha \in (\pi/6, \pi/4)$. Infact, as mentioned in Section 4 Extensions of [Tr] "this technical assumption guarantees that the orbit starting at the endpoints of L (the leg of P) are simple periodic loops." The proof in [Tr] used an counting argument which showed that even if there are generalized diagonals the number of nonexceptional strips, i.e. those which do not contain a point central symmetry, is still two.

Here I give a much simple proof which holds for all irrational right triangles for all (nonsimple) directions except for directions parallel to the hypotenuse. Note that if the direction is parallel to the hypotenuse the invariant surface breaks into two invariant sets, the positive levels and the negative levels. They are separated by a generalized diagonal which is the hypotenuse. In particular the above picture breaks down.

Fix a nonsimple direction ϕ which is not parallel to a side of the rhombus. Suppose that there are at least two distinct forward escape orbits on S_ϕ. Let $x_1 := (s_1, \phi)$ and $x_2 := (s_2, \phi)$ be a point on each of these orbits when it passes the last time through rhombus R_0. Suppose $s_1 < s_2$. Since almost every point is angularly recurrent there is a nonsingular point $x_3 := (s_3, \phi) \in R_0$ with $s_1 < s_3 < s_2$ whose orbit returns to R_0, i.e. there is a $M_3 > 0$ such that the angle of $T^{M_3} x_3$ is ϕ. Fix $N > 0$ sufficiently large. Since x_1 and x_2 are forward escape orbits we can find positive integers M_1, M_2 such that $T^{M_1} x_1 \in R_N$ and $T^{M_2} x_2 \in R_N$. Remark that if the orbit segment $\{x_1, Tx_1, \ldots, T^{M_1} x_1\}$, is singular, one can replace x_1 by a close point x_1' whose orbit segment has the same labels at the orbit segment of x_1. The same holds for x_2. In particular the orbit segment of x_i' starts on level 0 and arrives at time M_i to level N without revisiting level 0. This remark allows us to assume without loss of generality that the orbits of the x_i are nonsingular. Let $M := \max\{M_1, M_2, M_3\}$. Fix $\varepsilon > 0$ such that T^M is continuous when restricted to the ε-ball around x_i, $i = 1, 2, 3$. By this continuity there is a simple directions θ close to ϕ such that on the surface S_θ there are at least two exception strips of the form $E^+_{0,N}$, a contradiction. Thus there is a unique forward escape orbit on S_ϕ. Similarly there is a unique backward escape orbit. □

Proof of Theorem 6. Simply repeat the proof of Theorem 3 of [Tr]. The fact that the (implicit) assumption of this proof is verified is exactly the result of Theorem 5. □

Proof of Lemma 7. Consider the right triangle and its related rhombus so that the longer leg is horizontal. Suppose that the angle α between the hypotenuse and this leg satisfies $2n\alpha < \frac{\pi}{2}$ and $(2n+1)\alpha > \frac{\pi}{2}$ for some $n \geq 1$. Then the vertical orbit g starting at the left endpoint of this leg hits the leg again perpendicularly at an interior point (the case $n = 1$, $\alpha \in (\frac{\pi}{6}, \frac{\pi}{4})$ is illustrated in Figure 3a). Thus this leg is end point good for triangles for which $\alpha \in \cup_{n \geq 1} (\frac{\pi}{4n+2}, \frac{\pi}{4n})$.

Next consider the orbits perpendicular to the hypotenuse. If the angle α satisfies $2n\alpha > \frac{\pi}{2}$ and $(2n-1)\alpha < \frac{\pi}{2}$ for some $n \geq 2$ then the perpendicular orbits starting at the endpoints of the hypotenuse hit the hypotenuse perpendicularly again at an interior point (the case $n = 2$ is illustrated in Figure 3b). Thus the hypotenuse is end point good for right triangles for which $\alpha \in \cup_{n \geq 2} (\frac{\pi}{4n}, \frac{\pi}{4n-2})$. □

Proof of Lemma 9. Think of S_θ as being tiled by copies of P which can be enumerated P_i. Fix i and consider $P_i \cap \{O_j\}$. This set is an at most countable collection of parallel line segments $U_{i,j}$. Let $U_i = \cup_j U_{i,j}$. Consider the projection $\pi : S_\theta \to P$. This is a countable to one map. Enumerate the points in $\pi^{-1}(p)$: $(p_1, \theta_1), (p_2, \theta_2), \ldots$. For each $p \in P$ the set of directions $\{\theta_k\}$ is dense in $\mathbb{S}^1$. By assumption each of these directions is the direction of a periodic orbit through p, unless p_k is

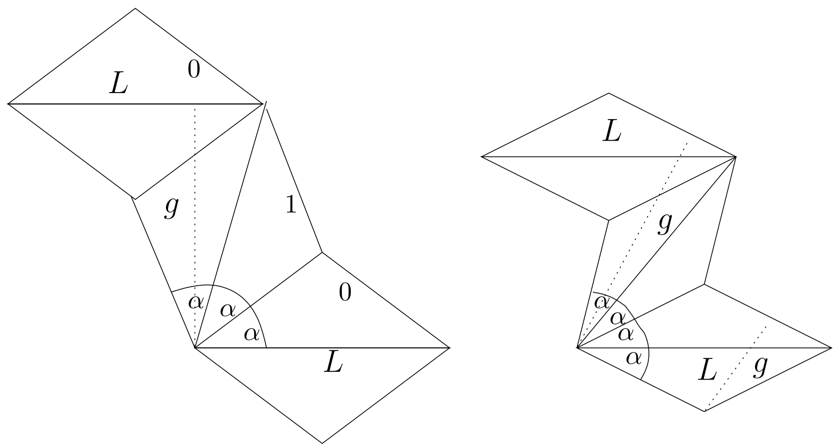

FIGURE 3. g hits perpendicularly at an interior point.

in some $U_{i,j}$. If the periodic directions through p are not dense then there exists infinitely many U_i such that $p \in \pi(U_i)$. Suppose there exists i, i' such that $p \in \pi(U_i) \cap \pi(U_{i'})$. Since the line segments in $\pi(U_i)$ and $\pi(U_{i'})$ are transverse this intersection is at most countable. One completes the proof by taking a union over i, i'. □

Proof of Proposition 3. In Veech polygons there is a dense set of periodic directions, and each periodic direction is completely foliated by periodic orbits except for a finite number of generalized diagonals. Apply the argument of Lemma 9 to finish the proof. □

Proof of Proposition 4. Consider the square, its invariant surface is a torus tiled by four squares. Fix a rational direction. Every point in this direction is periodic except for generalized diagonals. Suppose that the square unfolds to a torus. Consider the tiling of the plane by squares representing the torus. The intersection points of two generalized diagonals have rational coordinates. Thus we conclude that any point in the square for which at least one coordinate is irrational lies in at most one generalized diagonal and thus satisfies the conclusions of the proposition.

Consider a rational point $(p_1/q, p_2/q)$ with $gcd(p_1, p_2, q) = 1$. Tile the plane by $(1/q, 1/q)$ squares. Consider the orbits of slope a/b with $gcd(a, b) = 1$ starting at the point (p_1, p_2). Such a orbit either correspond to a periodic generalized diagonal or a periodic billiard orbit. It is a generalized diagonal if and only if there exists an $i \in \mathbb{Z}$ such that $q|(p_1 + ia)$ and $q|(p_2 + ib)$ (see Figure 4). Note that $p_i \in \{0, 1, \ldots, q\}$ and not being in the set $V(P)$ is equivalent to at least one of the p_i differing from 0 or q. If $p_1 \notin \{0, q\}$ and $q|a$ then q never divides $p_1 + ia$ and thus applying the above condition yields that the orbit is periodic. Thus the periodic directions through the given point include the set

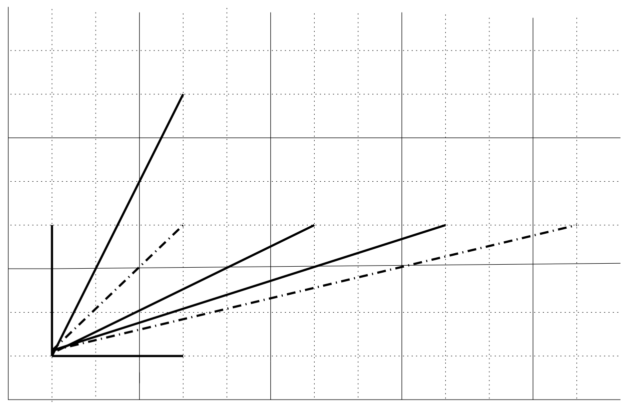

FIGURE 4. Solid lines are periodic directions while dashed lines are generalized diagonals starting at the point $(1/3, 1/3)$.

$\{qa'/b : qa' \wedge b = 1\}$. Similarly applying the condition in the case $p_2 \notin \{0, q\}$ and $q|b$ yields that the periodic directions include the set $\{a/qb' : a \wedge qb' = 1\}$. Both of these sets of directions are dense in $\mathbb{S}^1$.

Now if P is arithmetic then the unfolded surface is a torus cover. The periodic orbits constructed above for the square lift to periodic orbits in P. $\square$

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